

## 2022 John O'Bryan Team Competition Key

1. For all **natural numbers**  $n$ , let  $S(n)$  be the sum of the digits of  $n$  plus the number of digits of  $n$ . For instance,  $S(125) = 1 + 2 + 5 + 3 = 11$ . Note that the first digit of  $n$ , when reading from left to right, cannot be zero.
- Determine  $S(12408)$
  - Determine all numbers  $m$  such that  $S(m) = 4$
  - Determine whether or not there exists a natural number  $m$  such that  $S(m) - S(m + 1) > 50$ . Provide a clear justification for your answer.

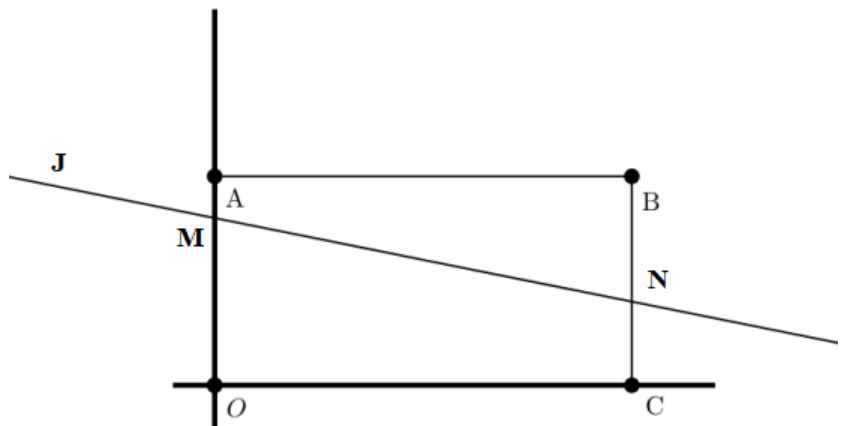
### Problem 1 Solution:

- $S(12408) = 1 + 2 + 4 + 0 + 8 + 5 = 20$ .**
- Consider 1-digit numbers, then  $m = 3$ . Among 2-digit numbers we need those with the sum of their digits equal to 2; so we have  $m = 11$  and  $m = 20$ . Among 3-digit numbers we need those with the sum of their digits equal to 1, so we have  $m = 100$ . For numbers with 4 or greater than 4 digits,  $S(m) > 4$ , so only one-, two, and three-digit numbers are possible. **The answer is 3, 11, 20, 100.**
- If  $m$  and  $m + 1$  differ only in one last digit the inequality  $S(m) - S(m + 1) > 50$  is not possible. However, it is possible to satisfy the inequality. Consider the case when  $m$  is a  $k$ -digit number composed of  $k$  9's. Then  $S(m) = 9k + k = 10k$  and  $S(m + 1) = 1 + (k + 1) = k + 2$ . If  $k \geq 6$ , therefore, we have  $S(m) - S(m + 1) = 9k - 2 > 50$ . For  $k = 6$ , for instance, we obtain  $m = 999,999$  and  $S(999,999) - S(1,000,000) = 60 - 8 > 50$ . **The answer is YES, for example, if  $m = 999,999$ .**

2. Suppose line J in the xy-plane is given by the equation  $5y + (2c - 4)x - 10c = 0$ , where  $c$  is some real number. Furthermore, suppose line J intersects the rectangle with vertices  $O(0,0)$ ,  $A(0,6)$ ,  $B(10,6)$ , and  $C(10,0)$  at point M on line segment OA and point N on line segment BC.
- Show that  $1 \leq c \leq 3$
  - Show that the area of quadrilateral AMNB is one-third the area of rectangle OABC
  - Find the equation, in terms of  $c$  of the line parallel to J that has the following characteristics: (1) the line intersects segment OA at point R, (2) the line intersects segment BC at point Q, and (3) quadrilaterals AMNB, MNQR, and RQCO all have the same area.

**Problem 2 Solution:**

- a. Since M is on OA, the x-coordinate of M is 0. The y-coordinate M is then the solution to the equation  $5y - 10c = 0$ . So,  $y = 2c$  and, therefore, line J intersects OA at point  $M(0, 2c)$ . For M to be on segment OA,  $0 \leq 2c \leq 6$ , which implies that  $0 \leq c \leq 3$ . Similarly, the x-coordinate of N is 10, so the y-coordinate is the solution to  $5y + (2c - 4)(10) - 10c = 0$ , which has solution is  $y = 8 - 2c$ . Hence,  $0 \leq 8 - 2c \leq 6$  or  $1 \leq c \leq 4$ . **Thus  $0 \leq c \leq 3$  and  $1 \leq c \leq 4$  must both be true,  $1 \leq c \leq 3$ .**



- b. Observe AMNB is a trapezoid with base AB and parallel sides are AM and BN, so has area  $\frac{AB(AM+BN)}{2} = \frac{10((6 - 2c) + (6 - (8 - 2c)))}{2} = \frac{10(4)}{2} = 20$ . The area of rectangle OABC =  $10(6) = 60$ , so **the area of trapezoid AMNB = (1/3)(area of rectangle OABC).**
- c. In order for the quadrilaterals to have equal area, it is sufficient to demand that the area of trapezoid RQCO has area 20 (i.e. 1/3 the area of OABC). Let point  $X(5, b)$  be the midpoint of segment PQ. Then the average of the y-coordinates of R and Q is  $b$ , so the area of RQCO is  $b(10) = 10b$ , so  $b = 2$ . Hence the point  $X(5, 2)$  is on the desired line. The slope of this line is the same as line J, so it is given by  $\frac{4 - 2c}{5}$ . **Thus, the equation of the line is  $y = \left(\frac{4 - 2c}{5}\right)x + (2c - 2)$ .**

3. Suppose  $g(x)$  is the quadratic function  $g(x) = x^2 - ax + b$ , where  $a$  and  $b$  are **natural numbers**.
- If  $a = b = 2$ , find the set of real roots of the expression  $g(x) - x$ .
  - If  $a = b = 2$ , find the set of real roots of the expression  $g(g(x)) - x$ .
  - Find the number of pairs of **natural numbers**  $(a, b)$  where  $1 \leq a \leq 2022$ ,  $1 \leq b \leq 2022$ , and every root of the expression  $g(g(x)) - x$  is an integer.

### Problem 3 Solution

- If  $a = 2$  and  $b = 2$ , then  $g(x) = x^2 - 2x + 2$ . Hence,  $g(x) - x = x^2 - 3x + 2 = (x-2)(x-1)$ . **Therefore, the roots of  $g(x) - x$  are 1 and 2.**
- We now determine  $g(g(x)) - x$ . Note that  $g(g(x)) = (x^2 - 2x + 2)^2 - 2(x^2 - 2x + 2) + 2 = x^4 - 4x^3 + 6x^2 - 4x + 2$ . Therefore,  $g(g(x)) - x = x^4 - 4x^3 + 6x^2 - 5x + 2$ . Note that 1 is a root of  $g(g(x)) - x$ . Then,  $x^4 - 4x^3 + 6x^2 - 5x + 2 = (x-1)(x^3 - 3x^2 + 3x - 2) = (x-1)(x-2)(x^2 - x + 1)$ . Note that  $x^2 - x + 1$  has no real roots since its discriminant is  $1^2 - 4 \cdot 1 \cdot 1 = -3 < 0$ . **Therefore, the only real roots of  $g(g(x)) - x$  are 1 and 2.**
- The answer is 43.** First, we claim that if  $r$  is a root of  $g(x) - x$ , then  $r$  is a root of  $g(g(x)) - x$ . Since  $r$  is a root of  $g(x) - x$ ,  $g(r) - r = 0$ , or  $g(r) = r$ . Therefore,  $g(g(r)) - r = g(r) - r = 0$ . Hence, any root of  $g(x) - x$  is a root of  $g(g(x)) - x$  and  $g(x) - x$  is a factor of  $g(g(x)) - x$ .

Note that  $g(g(x)) - x = g(x^2 - ax + b) - x = (x^2 - ax + b)^2 - a(x^2 - ax + b) + b - x = x^4 - 2ax^3 + (a^2 + 2b - a)x^2 - (2ab - a^2 + 1)x + (b^2 - ab + b)$ . Since  $g(x) - x = x^2 - (a+1)x + b$ ,  $g(g(x)) - x$  can be factored into  $(x^2 - (a+1)x + b)(x^2 - (a-1)x + (b-a+1))$ . Since both factors are *monic* (univariate polynomials), every root of  $g(g(x)) - x$  is an integer if and only if the discriminants of both of these quadratic factors are perfect squares. These two discriminants are  $(a+1)^2 - 4b = a^2 + 2a + 1 - 4b$  and  $(a-1)^2 - 4(b-a+1) = a^2 + 2a + 1 - 4b - 4$ . Note that the first discriminant is four more than the second discriminant.

The only two perfect squares that differ by 4 are 4 and 0. This statement is true since if  $r, s$  are non-negative integers such that  $r^2 - s^2 = 4$ , then  $(r-s)(r+s) = 4$ . Since  $r, s$  are non-negative,  $(r-s, r+s) = (2, 2)$  or  $(1, 4)$ . In the latter case,  $r - s = 1$  and  $r + s = 4$ . Therefore,  $r = 5/2$  and  $s = 3/2$ , which are not integers. Therefore,  $(r - s, r + s) = (2, 2)$ , i.e.  $r = 2, s = 0$ . Hence, the larger perfect square is  $2^2 = 4$  and the smaller perfect square is 0. Therefore,  $a^2 + 2a + 1 - 4b = 4$ . Rearranging this and factoring yields  $(a+1)^2 = 4(b+1)$ . Since  $(a+1)^2$  and 4 are perfect squares,  $b+1$  is a perfect square. Therefore, there exists a positive integer  $m$  such that  $b+1 = m^2$ . Then  $b = m^2 - 1$ . Consequently,  $(a+1)^2 = 4m^2$ . Since  $a$  is a natural number,  $a+1 = 2m$ . Hence,  $a = 2m - 1$ . Therefore,  $(a, b) = (2m - 1, m^2 - 1)$ .

We now verify that all such  $(a, b)$  have the property that the roots of  $x^2 - (a+1)x + b$  and  $x^2 - (a-1)x + (b-a+1)$  are all integers, implying that every root of  $g(g(x)) - x$  is an integer. Substituting  $(a, b) = (2m - 1, m^2 - 1)$  into these two polynomials yield  $x^2 - 2mx + m^2 - 1 = (x - (m-1))(x - (m+1))$  and  $x^2 - (2m-2)x + (m^2 - 2m + 1) = (x - (m-1))(x - (m-1))$ . Since  $m$  is a positive integer, all four roots of  $g(g(x)) - x$  are integers.

Since  $1 \leq a, b \leq 2022$ , it remains to find the number of positive integers  $m$  such that  $1 \leq 2m - 1, m^2 - 1 \leq 2022$ . Since  $1 \leq m^2 - 1 \leq 2022, 2 \leq m^2 \leq 2023$ . Hence,  $2 \leq m \leq \lfloor \sqrt{2023} \rfloor = 44$ , where  $\lfloor t \rfloor$  denotes the largest integer less than or equal to  $t$ . There are 43 solutions for  $m$ , namely  $m = 2, 3, \dots, 44$ . These values of  $m$  clearly satisfy  $1 \leq 2m - 1 \leq 2022$ . Therefore, the number of ordered positive integer pairs  $(a, b)$  that results in  $g(g(x)) - x$  having all integer roots is 43.

4. Jaden takes a mathematics test consisting of 100 questions, where the answer to each question is either TRUE or FALSE. For every five consecutive questions on the test, the answers to exactly three of the questions are TRUE. If the answers to Question 1 and Question 100 are both FALSE:
- Find the number of questions on the test for which the correct answer is TRUE.
  - Find the correct answer to the sixth question on the test.
  - Explain how Jaden can correctly answer ALL 100 questions on the test.

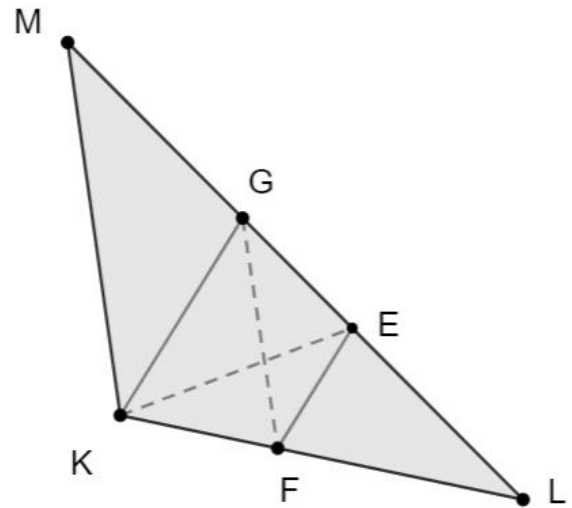
**Problem 4 Solution:**

- The answer is 60.** Split the 100 problems into groups of 5, namely 1–5, 6–10, 11–15, . . . , 91–95, 96–100. Since there are 100 problems and five problems per group and every set of five consecutive problems contain exactly three problems whose answer is TRUE, each group contains three problems whose answers are TRUE. **Since there are 20 groups, there are  $20 \times 3 = 60$  problems whose answers are TRUE on the test.**
- Consider the problems 1, 2, 3, 4, 5, 6. Among problems 1–5, there are exactly three problems whose answer is TRUE. Since the answer to the first problem is FALSE, among problems 2 – 5, exactly three of these problems have answer TRUE. Now consider problem 6. **Since problems 2 – 6 contains exactly three problems whose answers are TRUE and problems 2–5 already contain 3 such problems, the answer to problem 6 is FALSE.**
- The answer to the question depends upon the fact that the answer to problem  $n$  has the same answer as problem  $n + 5$ . Consider the problems  $n, n + 1, n + 2, n + 3, n + 4, n + 5$ . Note that problems  $n, n+1, n+2, n+3, n+4$  contain three problems whose answers are TRUE and problems  $n + 1, n + 2, n + 3, n + 4, n + 5$  contain three problems whose answers are TRUE. Note that problems  $n + 1, n + 2, n + 3, n + 4$  contain either 2 or 3 problems whose answers are TRUE. In the former case, the answers to both problem  $n$  and problem  $n + 5$  are TRUE. In the latter case, the answers to both problem  $n$  and problem  $n+ 5$  are FALSE. In either case, problems  $n$  and  $n + 5$  have the same answer.

Using this fact, problems  $\{1, 6, 11, 16, \dots, 91, 96\}$  have the same answers. So do  $\{2, 7, 12, 17, \dots, 92, 97\}$ ,  $\{3, 8, 13, 18, \dots, 93, 98\}$ ,  $\{4, 9, 14, 19, \dots, 94, 99\}$  and  $\{5, 10, 15, 20, \dots, 95, 100\}$ . For each of these five groups of problems, if we can determine the answer to one problem in the group, we can determine the answers to every problem in the group. Since the answer to problem 1 is FALSE, the answers to problems  $\{1, 6, 11, 16, \dots, 91, 96\}$  are all FALSE. Since problem 100 is FALSE, then the answers to problems  $\{5, 10, 15, 20, \dots, 95, 100\}$  are also FALSE. Since problems 1 and 5 have answers FALSE, and exactly three of problems 1, 2, 3, 4, 5 have answer TRUE, problems 2, 3, 4 have answer TRUE. Therefore, the answers to the remaining problems  $\{2, 7, 12, 17, \dots, 92, 97\}$ ,  $\{3, 8, 13, 18, \dots, 93, 98\}$ ,  $\{4, 9, 14, 19, \dots, 94, 99\}$  are all TRUE. **Having determined the correct answer to each question, Jaden achieves a perfect score by answering FALSE, TRUE, TRUE, TRUE, FALSE to the first five questions, and repeating this pattern for each block of five consecutive questions.**

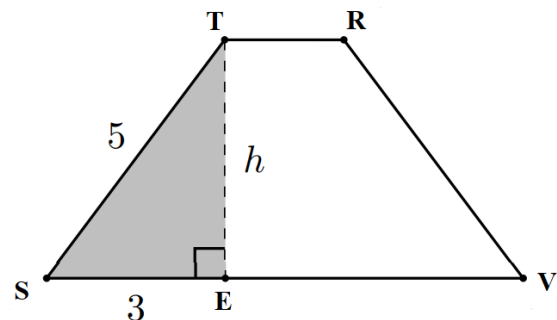
5. Suppose quadrilateral STRV is an isosceles trapezoid, with  $ST = 5$  cm,  $RV = 5$  cm,  $TR = 2$  cm, and  $SV = 8$  cm.

- What is the length of diagonal SR?
- For the isosceles trapezoid in part (a), what is the exact value of the **cosine** of  $\angle RTS$ ?
- In triangle KLM below, points G and E are points on segment LM so that  $\angle MKG \cong \angle GKE \cong \angle EKL$ . Also, point F is located on segment KL so that segment GF is parallel to segment KM. If quadrilateral KFE G is an isosceles trapezoid and the measure of  $\angle KLM$  is  $84^\circ$ , find the measure of  $\angle MKG$ .

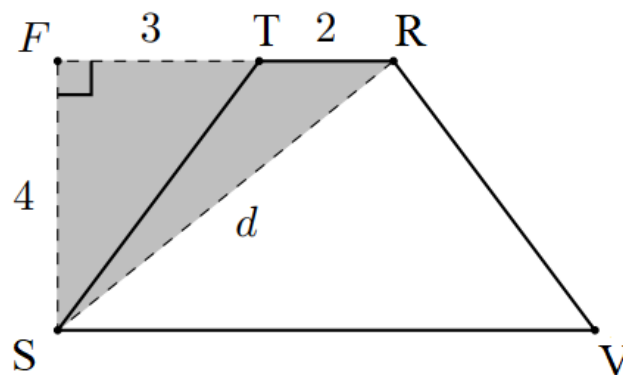


**Problem 5 Solution:**

- Let TE be the altitude of the trapezoid, so that angle TES is a right triangle with hypotenuse  $ST = 5$  and  $SE = \frac{8-2}{2} = 3$ . Therefore, the altitude is 4 (the sides of triangle TES form the Pythagorean triple 3-4-5).



Using the fact that the altitude of the trapezoid is 4, construct altitude SF from point S. **From the right triangle SFR, where  $SF = 4$ ,  $FR = FT + TR = 3 + 2 = 5$ , we find  $SR = \sqrt{16 + 25} = \sqrt{41}$  units.**

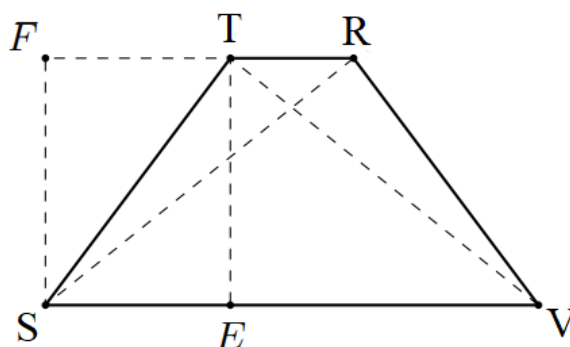


- b. Using triangle STR and the Law of Cosines,  $41 = 4 + 25 - 20 \cos \angle RTS$ .

So,  
 $\cos \angle RTS = -0.6$ .

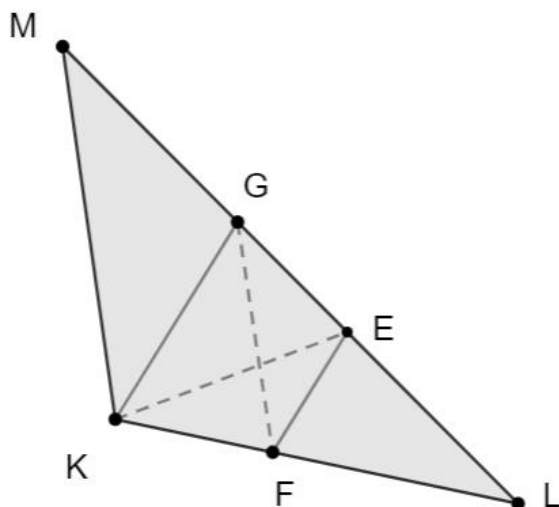
As an alternative approach, since  $FT = SE = 3$ , and  $ST = 5$ ,  $\cos \angle FTS = 3/5$ .

Then  $\cos \angle RTS = \cos(180 - \angle FTS) = -\cos \angle FTS = -3/5 = -0.6$ .



- c. Let  $m\angle MKG = x$ . Since segments KM and FG are parallel, quadrilateral KFEG is an isosceles trapezoid. We have that  $m\angle KML = m\angle FGE = m\angle FKE$ , so  $m\angle LKM = 3(m\angle KML)$ . Since the two angles sum up to 96 degrees, we have that  $m\angle LKM = 72^\circ = 3x$ . Thus,  $x = 24^\circ$ .

As an alternative explanation, since quadrilateral KFEG is an isosceles trapezoid, due to symmetry, triangle KLG is an isosceles triangle. Then  $2x + x + 84^\circ = 180^\circ$  and  $x = 24^\circ$



6. If  $M$  is a **natural number**, then a “nice division” of  $M$  is a partition of the set  $\{1, 2, \dots, M\}$  into two disjoint, non-empty subsets  $A_1$  and  $A_2$  such that the **sum** of the numbers in  $A_1$  is equal to the **product** of the numbers in  $A_2$ . If  $M = 8$ , for instance, then  $A_1 = \{2, 4, 5, 6, 7\}$  and  $A_2 = \{1, 3, 8\}$  is a “nice division” of  $M$ .
- Find a “nice division” of  $M = 7$ .
  - Find a value of  $M$  such that there are two distinct “nice divisions” of  $M$ .
  - A curious student claims that for every natural number  $M \geq 5$ , there is a “nice division” of  $M$ . Show or explain why the student is correct.

**Problem 6 Solution:**

- Let  $S_1 = \{2, 4, 5, 7\}$  and  $S_2 = \{1, 3, 6\}$ . Note that  $2 + 4 + 5 + 7 = (1)(3)(6) = 18$ .
- Consider taking  $S_2 = \{1, x, y\}$  for some  $1 < x < y \leq M$ , and  $S_1$  the complement of  $S_2$  in  $\{1, 2, \dots, M\}$ . This is a nice division if and only if the sum of the values in  $S_1$  equals the product of the values in  $S_2$ . So:

$$\frac{M^2+M}{2} - 1 - x - y = xy \text{ or } \frac{M(M+1)}{2} = (x+1)(y+1)$$

Similarly, let  $S_2 = \{x', y'\}$ . Then, we have a nice division if and only if

$$\frac{M^2+M}{2} - x' - y' = x'y' \text{ or } \frac{M(M+1)}{2} + 1 = (x'+1)(y'+1)$$

With  $M = 10$ , we have  $(x+1)(y+1) = 55 = 5 \times 11$  and  $(x'+1)(y'+1) = 56 = 7 \times 8$ . Thus,  $(x, y) = (4, 10)$  and  $(x', y') = (6, 7)$ , and we get two distinct nice divisions of 10:  $S_1 = \{2, 3, 5, 6, 7, 8, 9\}$ ,  $S_2 = \{1, 4, 10\}$ ; and  $S_1 = \{1, 2, 3, 4, 5, 8, 9, 10\}$ ,  $S_2 = \{6, 7\}$ .

- Use the process and equation from part (b) for arbitrary natural number  $M \geq 5$ . That is, find  $x$  and  $y$  values that satisfy the equation  $\frac{M(M+1)}{2} = (x+1)(y+1)$

If  $M \geq 6$  and is even, then let  $x = \frac{M-2}{2}$  and  $y = M$ . These satisfy the equation and  $1 < x < y \leq M$ , so that  $x$  and  $y$  are both elements in  $\{1, 2, \dots, M\}$ .

If  $M \geq 5$  and is odd, then let  $x = \frac{M-1}{2}$  and  $y = M - 1$ . These satisfy the equation and  $1 < x < y \leq M$ , so that  $x$  and  $y$  are both elements in  $\{1, 2, \dots, M\}$ .

Thus, all  $M \geq 5$  admit a nice division.